Langevin equations with multiplicative noise: Resolution of time discretization ambiguities for equilibrium systems

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A Langevin equation with multiplicative noise is an equation schematically of the form $d\mathbf{q}/dt = -\mathbf{F}(\mathbf{q}) + e(\mathbf{q})\boldsymbol{\xi}$, where $e(\mathbf{q})\boldsymbol{\xi}$ is Gaussian white noise whose amplitude $e(\mathbf{q})$ depends on $\mathbf{q}$ itself. Such equations are ambiguous, and depend on the details of one's convention for discretizing time when solving them. I show that these ambiguities are uniquely resolved if the system has a known equilibrium distribution $\exp[-V(\mathbf{q})/T]$ and if, at some more fundamental level, the physics of the system is reversible. I also discuss a simple example where this happens, which is the small frequency limit of Newton's equation $\ddot{\mathbf{q}} + e^2(\mathbf{q})\dot{\mathbf{q}} = -\boldsymbol{\nabla}V(\mathbf{q}) + e^{-1}(\mathbf{q})\boldsymbol{\xi}$ with noise and a $\mathbf{q}$-dependent damping term. The resolution does *not* correspond to simply interpreting naive continuum equations in a standard convention, such as Stratonovich or Itô.

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I. INTRODUCTION

A Langevin equation of the form

$$\sigma\dot{\mathbf{q}} = -\boldsymbol{\nabla}V(\mathbf{q}) + \boldsymbol{\zeta}, \tag{1a}$$

$$\langle \zeta_i(t)\zeta_j(t')\rangle = 2\sigma T\delta_{ij}\delta(t-t'), \tag{1b}$$

can appear as the effective description of highly overdamped motion of some coordinates $\mathbf{q}$ in a potential V in contact with a thermal bath. The damping, and the random force term, arise from interactions with the thermal bath. By overdamped, I mean that at a slightly more fundamental level, the equation of motion might be, for example,

$$\ddot{\mathbf{q}} + \sigma\dot{\mathbf{q}} = -\boldsymbol{\nabla}V(\mathbf{q}) + \boldsymbol{\zeta}, \tag{2}$$

with $\boldsymbol{\zeta}$ as before. Here, σ is a damping coefficient, and $\boldsymbol{\zeta}$ is a Gaussian thermal noise term. In the limit that σ is large compared to the inverse time scales of interest, one can ignore the $\ddot{\mathbf{q}}$ term and so obtain Eq. (1a).

In this paper, I want to consider the case where (a) the damping coefficient σ depends on $\mathbf{q}$ itself,

$$\sigma_{ij}(\mathbf{q})\dot{q}_i = -\nabla_i V(\mathbf{q}) + \zeta_i, \tag{3a}$$

$$\langle \zeta_i(t)\zeta_j(t')\rangle = 2\sigma_{ij}(\mathbf{q})T\delta(t-t'), \tag{3b}$$

and (b) it is already known through other means that $V(\mathbf{q})$ is an effective potential that gives the equilibrium distribution of $\mathbf{q}$ as

$$P_{\text{eq}}(\mathbf{q}) = \exp[-V(\mathbf{q})/T] \tag{4}$$

in whatever approximation may be relevant to the problem of interest. Equation (3) is a special case of what are known as Langevin equations with multiplicative noise. In general, such equations are notorious for being ambiguous—they are sensitive to exactly how one discretizes time. The purpose of this paper is to show that, with a few very general assumptions, property (4) is sufficient to uniquely resolve this ambiguity. Moreover, this resolution does *not* correspond to any standard interpretation (Itô or Stratonovich) of the Langevin equation (3).

Though my arguments will be somewhat general, it helps to have in mind a concrete example of an unambiguous description for which Eq. (3) will be a limiting case. A simple example is the analog of Eq. (2) with $\mathbf{q}$-dependent damping:

$$\ddot{q}_i + \sigma_{ij}(\mathbf{q})\dot{q}_j = -\nabla_i V(\mathbf{q}) + \zeta_i, \tag{5}$$

with the noise again Eq. (3b). As I discuss in Appendix A, this equation is unambiguous and has equilibrium distribution (4).

I also mention the application of interest to me personally, which motivated this work [1]: electroweak baryon number violation in the early universe. Its study requires understanding the effective dynamics of fluctuations in weakly coupled high-temperature non-Abelian gauge theories, where there is an effective theory of the form (3) at the relevant distance and time scales, but where it is much more straightforward to analyze static issues, such as Eq. (4), than the subtleties of dynamical ones.

II. A FIRST PASS

Let's rewrite Eq. (3) in the equivalent form

$$\dot{\mathbf{q}} = -\mathbf{F}(\mathbf{q}) + e(\mathbf{q})\boldsymbol{\xi}_i, \tag{6a}$$

$$\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t-t'), \tag{6b}$$

where the $\mathbf{q}$ dependence has been scaled out of the noise by defining $\boldsymbol{\xi} \equiv e(\mathbf{q})\boldsymbol{\zeta}$, and the matrix e and vector $\mathbf{F}$ are

$$e(\mathbf{q}) \equiv [\sigma(\mathbf{q})]^{-1/2}, \tag{7}$$

$$\mathbf{F}(\mathbf{q}) \equiv \sigma^{-1}(\mathbf{q})\boldsymbol{\nabla}V(\mathbf{q}). \tag{8}$$

Two standard conventions for discretizing time, and so removing the ambiguities inherent in equations like Eq. (3), are the Itô convention,[1]

$$\mathbf{q}_t-\mathbf{q}_{t-\Delta t}=-\Delta t\mathbf{F}^{\text{Itô}}(\mathbf{q}_{t-\Delta t})+e(\mathbf{q}_{t-\Delta t})\tilde{\boldsymbol{\xi}}_t, \tag{9}$$

and the Stratonovich convention,

$$\mathbf{q}_t-\mathbf{q}_{t-\Delta t}=-\Delta t\mathbf{F}^{\text{Strat}}(\overline{\mathbf{q}})+e(\overline{\mathbf{q}})\tilde{\boldsymbol{\xi}}_t, \tag{10}$$

$$\overline{\mathbf{q}}\equiv\frac{\mathbf{q}_t+\mathbf{q}_{t-\Delta t}}{2}, \tag{11}$$

where in both cases the discretized noise correlation is

$$\langle\tilde{\xi}_{it}\tilde{\xi}_{jt'}\rangle=2T\Delta t\,\delta_{ij}\delta_{tt'}. \tag{12}$$

The specification of $\mathbf{F}^{\text{Strat}}(\overline{\mathbf{q}})$ as opposed to $\mathbf{F}^{\text{Strat}}(\mathbf{q}_{t-\Delta t})$ in Eq. (10) is actually irrelevant: it is only the $\mathbf{q}$ used to evaluate e that causes the difference between these two conventions in the $\Delta t\rightarrow 0$ limit.

If one simply tries setting $\mathbf{F}^{\text{Itô}}=\mathbf{F}$ or $\mathbf{F}^{\text{Strat}}=\mathbf{F}$, the Itô and Stratonovich equations will give rise to different physics, and in particular different equilibrium distributions. The two conventions are identical in the $\Delta t\rightarrow 0$ limit only if one sets

$$F_i^{\text{Itô}}=F_i^{\text{Strat}}-Te_{ia,j}e_{ja}. \tag{13}$$

(Here and throughout, I adopt the notation that indices after a comma represent derivatives: $F_{i,j}\equiv\partial F_i/\partial q_j$ and $F_{i,jk}\equiv\partial^2F_i/\partial q_j\partial q_k$.)

The value of $\mathbf{F}$ that should be used for a particular physical problem therefore depends on what discretization convention one picks to use. Since the value of $\mathbf{F}$ is ambiguous, one approach might be to simply pick a discretization convention (it does not matter which one), and then choose $\mathbf{F}$ however necessary to reproduce the desired equilibrium distribution (4). To be concrete, let us choose Stratonovich convention. The Stratonovich Langevin equation (10) is well known to be equivalent to the following Fokker-Planck equation for the time evolution of the probability distribution $P(\mathbf{q},t)$:

$$\dot{P}=\nabla_i[Te_{ia}\nabla_j(e_{ja}P)+F_i^{\text{Strat}}P]. \tag{14}$$

My requirement is that this equation have the equilibrium distribution (4) as a time-independent solution. A simple way to achieve this is for

$$Te_{ia}\nabla_j(e_{ja}P_{\text{eq}})+F_i^{\text{Strat}}P_{\text{eq}}=0, \tag{15}$$

which gives

$$F_i^{\text{Strat}}=(\sigma^{-1})_{ij}\nabla_jV-Te_{ia}e_{ja,j}. \tag{16}$$

Compare to our naive starting point (8). Equivalently,

$$F_i^{\text{Itô}}=(\sigma^{-1})_{ij}\nabla_jV-T(\sigma^{-1})_{ij,j}. \tag{17}$$

So one might suspect that the Stratonovich equation (10) together with Eq. (16), or equivalently the Itô equation (9) together with Eq. (17), gives the correct description of the system. As we shall see, this is indeed the case.

Based on the presentation so far, the reader might be suspicious of two things. First, once we modify the equation, changing what we thought was $\mathbf{F}(\mathbf{q})$ to suit our needs, how do we know we are not supposed to change other things as well? In particular, what tells us that we shouldn't change $e(\mathbf{q})$ in some way, then make some compensating change in $\mathbf{F}$ to force the equilibrium distribution to work out? The basic answer is that $\mathbf{F}$ is sensitive to the details of short-time physics and regularization, whereas e is not, as evidenced by the fact that $\mathbf{F}$ must be changed when one adopts different conventions like Itô or Stratonovich, but e need not. This comes down to a discussion of the renormalizability of the theory, and its consequences for how, in principle, the theory should be matched to a more fundamental description of the physics. Such issues are more familiar in the context of theories defined by path integrals, and so much of the rest of this paper will be to translate the discussion into that language.

Second, the Stratonovich $\mathbf{F}$ of Eq. (16) is not the unique solution to Eq. (14) for $P=P_{\text{eq}}$. The general solution is

$$F_i^{\text{Strat}}=(\sigma^{-1})_{ij}\nabla_jV-Te_{ia}e_{ja,j}+h_ie^{+V/T}, \tag{18}$$

where $\mathbf{h}=\mathbf{h}(\mathbf{q})$ is any function with $\boldsymbol{\nabla}\cdot\mathbf{h}=0$.

I shall throughout focus on systems where the underlying physics, whatever it may be, is reversible. More specifically, I shall assume that the effective theory (1) must be defined in such a way that equilibrium time-dependent correlations, such as $\langle\mathbf{q}(t)\mathbf{q}(0)\rangle$, are invariant under time reversal. [As discussed in Appendix A, the behavior of the Langevin equation with inertia (5) has this property, even though the equation by itself is not time-reversal invariant.] As I will show, this assumption will rule out the extra term involving $\mathbf{h}$ in Eq. (18). I shall then discuss in more detail, in the language of path integrals, how the issue of whether e or $\mathbf{F}$ should be modified from their ''naive'' values is an issue of renormalizability. Specifically, I will define ''naive'' by assuming that there is some more fundamental description of the effective theory, such as the Langevin equation (5) with inertia, that is unambiguous and is described in terms of the same degrees of freedom $\mathbf{q}$. The ''naive'' values of e and $\mathbf{F}$ will simply be those defined by naively taking the low frequency limit of the more fundamental equation. I will show that terms associated with e require no ultraviolet renormalization, while terms associated with $\mathbf{F}$ do. Based on general procedures for matching effective theories to more fundamental underlying theories, one may then argue that $\mathbf{F}$, and not e, should be modified to make the physics work out right. This is something that, I believe, may be well known to the

[1]For a general review of background material for this paper, in notation close to what I use here, see, for example, Chap. 4 of Ref. [2]. The most substantial differences in notation are that my $\mathbf{F}$ and T are that reference's $\frac{1}{2}\mathbf{f}$ and $\frac{1}{2}\Omega$.

few people to whom it is well known; however, since there seems to be general confusion on this matter, it seems worthwhile to continue.

I should point out that the result (16) can be extracted from much older, general discussions of systems described by Langevin equations if one already knows the correct $e(\mathbf{q})$ to use (e.g., the naive one) and if one assumes that the underlying system is reversible. For example, this result can be derived by specializing the discussion of the review Ref. [3] if all the Langevin equations in that reference are interpreted using Stratonovich convention.

III. PATH INTEGRAL VERSION

The path integral corresponding to the Stratonovich Langevin equation (10) is, somewhat imprecisely [4],[2,3]

$$\mathcal{P}(\mathbf{q}'',\mathbf{q}',t''-t') = \int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} [d\mathbf{q}(t)] \exp\left[-\int_{t'}^{t''} dt\, L(\dot{\mathbf{q}},\mathbf{q}) \right], \tag{19}$$

$$\begin{aligned} L(\dot{\mathbf{q}},\mathbf{q}) = {} & \frac{1}{4T}(\dot{q}+F)_i \sigma_{ij} (\dot{q}+F)_j - \frac{1}{2} F_{i,i} \\ & + \frac{1}{2} e^{-1}_{ia} e_{ka,k} (\dot{q}+F)_i + \frac{T}{4} e_{ia,j} e_{ja,i} \\ & + \delta(0)\, \mathrm{tr} \ln e. \end{aligned} \tag{20}$$

Here and henceforth, I abbreviate $\mathbf{F}^{\mathrm{Strat}}$ as simply $\mathbf{F}$. The imprecision is just due to the fact that this path integral depends on the details of how time is discretized. I have implicitly assumed a time-symmetric discretization above. Specifically, Eq. (19) really means

$$\begin{aligned} \mathcal{P}(\mathbf{q}'',\mathbf{q}',t''-t') = {} & \lim_{\Delta t \to 0} N \int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \left[\prod_t d\mathbf{q}_t \right] \\ & \times \exp\left[-\Delta t \sum_t L\left(\frac{\mathbf{q}_t - \mathbf{q}_{t-\Delta t}}{\Delta t}, \frac{\mathbf{q}_t + \mathbf{q}_{t-\Delta t}}{2} \right) \right], \end{aligned} \tag{21}$$

where N is an overall normalization I shall not be explicit about, and where $\delta(0)$ in Eq. (20) means $(\Delta t)^{-1}$. If I had used some other discretization convention in the path integral, the Lagrangian L would be different from Eq. (20).

For my argument, it will be sufficient (and convenient) to focus on fluctuations about equilibrium. It is then enough to consider the $t' \to -\infty$ and $t'' \to +\infty$ limit of the path integral, writing

$$Z \equiv \int [d\mathbf{q}(t)] \exp\left[-\int_{-\infty}^{+\infty} dt\, L(\dot{\mathbf{q}},\mathbf{q}) \right]. \tag{22}$$

Because the system is dissipative, the boundary conditions on $\mathbf{q}$ at $t = \pm\infty$ decouple. Equilibrium correlation functions like $\langle \mathbf{q}(t)\mathbf{q}(0) \rangle$ can then be evaluated in the usual way as

$$\langle \cdots \rangle = Z^{-1} \int [d\mathbf{q}(t)] \exp\left[-\int_{-\infty}^{+\infty} dt\, L(\dot{\mathbf{q}},\mathbf{q}) \right] \cdots. \tag{23}$$

This form allows us to rewrite the Lagrangian in a form where the assumed time-reversal symmetry of correlations is manifest. Simply as an illustrative example, suppose for the moment that $\mathbf{F}$ *did* have the form (16) that I am trying to demonstrate. The Lagrangian (20) is not manifestly invariant under $t \to -t$. However, for the choice (16) of $\mathbf{F}$, this Lagrangian can be rewritten as

$$\begin{aligned} L(\dot{\mathbf{q}},\mathbf{q}) = {} & \frac{1}{4T} \dot{q}_i \sigma_{ij} \dot{q}_j + \frac{1}{4T} (\nabla_i V)(\sigma^{-1})_{ij} (\nabla_j V) \\ & - \frac{1}{2} \nabla_i [(\sigma^{-1})_{ij} \nabla_j V] + \frac{T}{4} (\sigma^{-1})_{ij,ij} + \delta(0)\, \mathrm{tr} \ln e \\ & + \frac{1}{2T} \partial_t V. \end{aligned} \tag{24}$$

Only the last term is not manifestly time-reversal invariant. However, with the path integral in the current form of Eq. (22), we are allowed to throw away terms in L that are total time derivatives. That is because they can be integrated and reexpressed in terms of the boundary conditions, which we know are irrelevant. [As reviewed in Appendix B, the fact that one may naively integrate total time derivatives is dependent upon the use of the symmetric time discretization (21).] The result is, that for the purpose of computing equilibrium (but time-dependent) correlators, we can replace L by

$$\begin{aligned} L_2(\dot{\mathbf{q}},\mathbf{q}) = {} & \frac{1}{4T} \dot{q}_i \sigma_{ij} \dot{q}_j + \frac{1}{4T} (\nabla_i V)(\sigma^{-1})_{ij} (\nabla_j V) \\ & - \frac{1}{2} \nabla_i [(\sigma^{-1})_{ij} \nabla_j V] + \frac{T}{4} (\sigma^{-1})_{ij,ij} + \delta(0)\, \mathrm{tr} \ln e. \end{aligned} \tag{25}$$

It is important to emphasize that L_2 cannot be used in the finite-time path integral (19): it would not produce the same $\mathcal{P}(\mathbf{q}'',\mathbf{q}',t''-t')$.

In passing, we can now see the problem if $\mathbf{F}$ contained the additional h term of Eq. (18). The Lagrangian L given by Eq. (20) would then produce additional terms in Eq. (24), one of which,

$$\frac{1}{2T} \dot{q}_i \sigma_{ij} h_j e^{V/T}, \tag{26}$$

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[2] There is a small change of notation from Ref. [4]: the $(\det e)^{-1}$ preexponential factor of that reference has been moved up into the exponent and absorbed into L.

[3] The formalism here is somewhat similar to that used to describe stochastic processes on curved manifolds, if one were to interpret σ_{ij} as a metric tensor g_{ij}. I should emphasize that this is *not* the problem I am studying. On a curved manifold, the desired equilibrium distribution would be $[\det \sigma]^{1/2} \exp(-V/T)$ instead of $\exp(-V/T)$. Mathematically, one can convert between these two problems by setting $V_{\mathrm{manifold}} = V + \frac{1}{2} T \ln \det \sigma$.

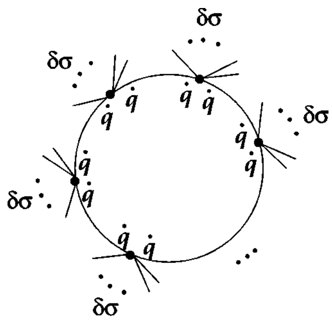

FIG. 1. The UV divergent one-loop graphs.

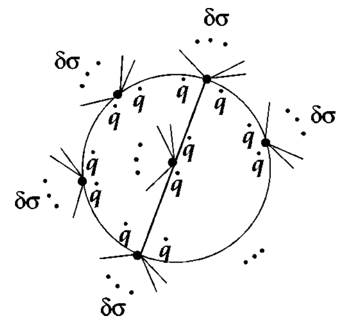

FIG. 2. The UV divergent two-loop graphs. In the vertices with three internal lines, the $\dot{\mathbf{q}}$'s can be associated with any two of them.

is odd under time reversal. So the appearance of h would contradict my assumption about reversibility, unless Eq. (26) can be waived away as a total time derivative. This would require

$$\delta F_i \equiv h_i e^{V/T} = (\sigma^{-1})_{ij} \nabla_j \phi \tag{27}$$

for some function $\phi(\mathbf{q})$. But, returning to Eq. (16), this just corresponds to a shift of $V \to V + \phi$, and the equilibrium distribution produced by the Fokker-Planck equation (14) would then be $\exp[-(V+\phi)/T]$ instead of the required $\exp(-V/T)$.

Now let us divorce ourselves from the particular form (16) that I've claimed for $\mathbf{F}$. Instead, just write down the most general form for a manifestly time-reversal invariant L_2:

$$L_2(\dot{\mathbf{q}},\mathbf{q}) = \frac{1}{4T} \dot{q}_i \sigma_{ij}(\mathbf{q}) \dot{q}_j + U(\mathbf{q}). \tag{28}$$

Before discussing how to determine the correct choices of σ_{ij} and U for this equation, I want to discuss its renormalization.[4] Imagine that the Lagrangian could be expanded in perturbation theory about some $\mathbf{q}_0$ with $\sigma(\mathbf{q}_0)$ finite and nonzero. For simplicity, say $\mathbf{q}_0 = 0$. The arguments that follow will be about the entire perturbative expansion, to all orders in perturbation theory. To this end, write

$$\frac{1}{4T} \dot{q}_i \sigma_{ij}(\mathbf{q}) \dot{q}_j = \frac{1}{4T} [\dot{q}_i \sigma_{ij}(0) \dot{q}_j + \dot{q}_i \delta\sigma_{ij}(\mathbf{q}) \dot{q}_j], \tag{29}$$

and formally consider $\delta\sigma$ as a perturbation.

I will now show that the $U(\mathbf{q})$ term in Eq. (28) requires counterterms, but the $\sigma(\mathbf{q})$ term does not. The Lagrangian (25) is super-renormalizable: counterterms are needed only for one loop and two loop diagrams. This can be seen by simple power counting of diagrams. The most ultraviolet (UV) divergent diagrams will be those whose vertices involve as many powers as possible of internal frequency. That is, vertices generated by the $\dot{q}_i \delta\sigma_{ij}(\mathbf{q}) \dot{q}_j$ terms in Eq. (25) will give more UV divergent behavior than vertices generated by the $U(\mathbf{q})$ terms. And the most divergent behavior will occur when the $\dot{\mathbf{q}}$'s are associated with internal rather than external lines. The most divergent one-loop diagrams are therefore those depicted by Fig. 1. The UV behavior of such graphs is

$$\int d\omega \frac{(\omega^2)^V}{(\omega^2)^I} = \int d\omega, \tag{30}$$

where $(\omega^2)^V$ are the powers of ω from the V vertices $\dot{\mathbf{q}}\delta\sigma\dot{\mathbf{q}}$, and $(\omega^2)^{-I}$ are the $\omega \to \infty$ behavior of the $I = V$ internal propagators. So this class of diagrams is linearly divergent in the UV. The linearly divergent piece is independent of external frequencies, and so requires a counterterm in $U(\mathbf{q})$ but not one in the kinetic term $\dot{\mathbf{q}}\sigma\dot{\mathbf{q}}$. That is, so far we have seen that the effective $U(\mathbf{q})$ is dependent on the details of short-time physics, but we have not seen any evidence that $\sigma(\mathbf{q})$ is.

Are there divergences associated with external frequency dependence? Expanding the diagrams of Fig. 1 in powers of external momenta, each such power will come at the expense of one of the factors of ω in Eq. (30). So no counterterms involving two external frequencies are necessary—that is, no counterterm for $\dot{\mathbf{q}}\sigma\dot{\mathbf{q}}$ is necessary. There could be a log divergence associated with one external frequency factor, except that the resulting ω integral $\int d\omega/\omega$ vanishes by the assumption that the equilibrium physics is time-reversal invariant (and the fact that I am using a symmetric discretization of the path integral, which respects that invariance).

Similarly, one could imagine a different method of hooking up the diagrams of Fig. 1 in which one or more of the $\dot{\mathbf{q}}$ factors were associated with external rather than internal lines. If this happens for two or more factors, the loop integral is no longer UV divergent and so is not sensitive to short-time physics. If it happens for only one factor, then the loop integral again vanishes by the assumption of time-reversal invariance. Therefore, the linear UV divergence (30) previously discussed is the *only* UV divergence at one loop.

Now consider two loop diagrams, again maximizing the divergence by using $\dot{\mathbf{q}}\delta\sigma\dot{\mathbf{q}}$ vertices rather than $U(\mathbf{q})$ ones, and associating all $\dot{\mathbf{q}}$'s with internal lines. The result is shown in Fig. 2. Simple power counting gives the maximum

[4] If the discussion in this paper is applied to a field theory, then I am assuming that the theory has already been regulated spatially, by the introduction of a small distance cutoff (either explicitly, by a spatial lattice cutoff, or implicitly by, for example, dimensional regularization), and that I am now separately considering the issue of regularizing and renormalizing small-time divergences. The discussion of divergences and power counting will therefore be somewhat different from what is usual in field theory, where one typically regulates space and time simultaneously.

possible new divergence at two loops as order

$$\int d^2\omega \frac{(\omega^2)^V}{(\omega^2)^I} = \int \frac{d^2\omega}{\omega^2}, \tag{31}$$

where $I = V+1$ and ω just counts powers of combinations of the two loop frequencies ω_1 and ω_2. A logarithmic divergence is possible, but this divergence has no dependence on external frequencies. As before, if we focus on the dependence of these diagrams on external frequency, then there is no sensitivity to short-time physics. If we go to yet higher orders in diagrams, there are no new divergences at all: naive power counting gives $\int \omega^{-2(L-1)} d^L\omega$ for an L-loop diagram.

The outcome of all this is that only $U(\mathbf{q})$ is sensitive to the details of short-time physics; $\dot{\mathbf{q}}\sigma\dot{\mathbf{q}}$ is not.

In the language of path integrals, the ''naive'' low-frequency limit of a theory is what you get if you simply replace the Lagrangian by its low-frequency limit. This replacement is naive when there are UV divergences in the resulting effective theory, which make the physics of that theory sensitive to how it is cut off in the UV—that is, sensitive to the details of the more fundamental theory. Fortunately, these details can be absorbed into a redefinition of the effective interactions which depend on them. In this case, that means $U(\mathbf{q})$ is modified from its naive form, but $\dot{\mathbf{q}}\sigma\dot{\mathbf{q}}$ is not.

Since this point is important, let me state it another way. The basic theory and technology of matching parameters of low-frequency effective theories to those of more fundamental theories, to any and all orders in perturbation theory, has been used in a number of problems in field theory, including Bose condensation [5], ultrarelativistic plasmas [6], heavy quark physics [7], and nonrelativistic plasma physics [8]. (For a general discussion, see also Ref. [9].) The basic idea is to fix a regularization for the effective theory, to then treat all the parameters of the effective theory as adjustable, to calculate low-frequency observables in both the effective theory and the more fundamental theory, and then to fix the parameters of the effective theory to obtain agreement. If we simply fixed $\sigma(\mathbf{q})$ and $U(\mathbf{q})$ to be their naive values, everything would match order by order in perturbation theory if it were not for the UV divergences in the effective theory: if diagrams only involved low frequencies in propagators and vertices, then each internal line and vertex in the underlying theory would match up with each one in the naive effective theory. The only thing we have to correct for are UV divergent graphs or subgraphs, where the correspondence would fail. But the previous discussion shows that such divergences have the form of counterterms for $U(\mathbf{q})$, and so the matching of the two theories can be fixed by appropriate adjustment of those counterterms.

If we had a specific underlying theory in mind, we could carry out this matching procedure by computing to two loops in perturbation theory. The crucial point here is that such a calculation is unnecessary, and detailed knowledge of the underlying theory is unnecessary, if we know that the equilibrium distribution is Eq. (4). We can then immediately solve the problem by simply requiring that we choose $U(\mathbf{q})$ so that the equilibrium distribution comes out right.

Returning to the Lagrangian (20) corresponding to the original Langevin equation, the only way we can change $U(\mathbf{q})$ without changing $\dot{\mathbf{q}}\sigma\dot{\mathbf{q}}$ is by changing $\mathbf{F}$, which justifies the argument for choosing $\mathbf{F}^{\rm Strat}$ given in Sec. II. Alternatively, one can uniquely determine $U(\mathbf{q})$ directly by requiring that the Euclidean Schrödinger equation, corresponding to the path integral with the generic Lagrangian L_2 [Eq. (28)], generate $\exp(-V/T)$ as its equilibrium distribution. This is carried out in Appendix C, which also explains some interesting distinctions between the interpretation of the Euclidean Schrödinger equation associated with L_2 [Eqs. (25) or (28)] and that associated with L [Eq. (20)]. The final result agrees with the initial analysis of Sec. II.

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APPENDIX A: THE LANGEVIN EQUATION WITH INERTIA

The Langevin equation (5) with inertia can be written in the form of a standard first-order Langevin equation by rewriting it as a system of first order equations, introducing $\mathbf{v} \equiv \dot{\mathbf{q}}$:[5]

$$\dot{\mathbf{v}} + \sigma(\mathbf{q})\mathbf{v} = -\boldsymbol{\nabla} V(\mathbf{q}) + e^{-1}(\mathbf{q})\boldsymbol{\xi}, \tag{A1a}$$

$$\dot{\mathbf{q}} = \mathbf{v}. \tag{A1b}$$

This equation is free of the usual discretization ambiguities. To see this, consider the difference between Stratonovich and Itô discretization conventions, which differ in how the $\mathbf{q}$ in the $e^{-1}(\mathbf{q})\boldsymbol{\xi}$ term is evaluated. That difference in $\mathbf{q}$ is $\frac{1}{2}\dot{\mathbf{q}}\Delta t$, which means a difference in the $e^{-1}(\mathbf{q})\boldsymbol{\xi}$ term of

$$\frac{\Delta t}{2}(e^{-1})_{ij,k}\dot{q}_k \xi_j . \tag{A2}$$

In a Langevin equation such as our effective theory (6), this would not vanish in the $\Delta t \to 0$ limit because $\boldsymbol{\xi}$ is order $(\Delta t)^{-1/2}$, and then so is $\dot{\mathbf{q}}$ by the equation of motion (6).[6] In the case at hand, matters are different, because $\dot{q}_k = v_k$, and, though Eq. (A1a) implies that the time derivative $\dot{\mathbf{v}}$ will be order $(\Delta t)^{-1/2}$, $\mathbf{v}$ itself is perfectly finite as $\Delta t \to 0$. So, in the case at hand, the difference (A2) vanishes in the $\Delta t \to 0$ limit.

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[5]Interestingly, this is a very special case of a general set of Langevin equations considered in Ref. [10].

[6]See, for example, Secs. 4.7–8 of Ref. [2] for a review.

[7]See, for example, Sec. 4.6 of Ref. [2] for a review, as well as Ref. [3].

It is also interesting to sketch how this works in the path integral formulation. Converting Eq. (5) directly into a path integral using standard methods[7] gives a path integral with Lagrangian

$$L=\frac{1}{4T}(\ddot{\mathbf{q}}+\sigma\dot{\mathbf{q}}+\boldsymbol{\nabla}V)\sigma^{-1}(\ddot{\mathbf{q}}+\sigma\dot{\mathbf{q}}+\boldsymbol{\nabla}V)+L_\eta+\delta(0)\,\mathrm{tr}\ln e, \tag{A3}$$

$$L_\eta=\bar{\eta}_i\{\delta_{ij}\partial_t^2+\sigma_{ij}\partial_t+(e^{-1})_{ai}[e_{ka,j}\ddot{q}_k+(e^{-1})_{ak,j}\dot{q}_k+\nabla_j(e_{ka}\nabla_k V)]\}\eta_j, \tag{A4}$$

where $\bar{\eta}$ and η are anticommuting Grassman fields (ghosts). A diagrammatic analysis similar to that discussed in the main text produces no UV divergences because the $\ddot{\mathbf{q}}\sigma^{-1}\ddot{\mathbf{q}}$ term means that $\mathbf{q}$ propagators go like ω^{-4} in the UV instead of only ω^{-2}. So diagrams converge faster in the UV than they did in the previous analysis.

Now let us check the equilibrium distribution for $\mathbf{q}$. Since Eq. (A1) formally has the same form as a generic first-order Langevin equation (6) [the number of degrees of freedom have simply doubled from $\mathbf{q}$ to $(\mathbf{q},\mathbf{v})$], we can use result (14) for the Fokker-Planck equation for the probability distribution $P(\mathbf{q},\mathbf{v},t)$, which translates in the present case to

$$\begin{aligned}\dot{P}&=\nabla_{v_i}\{T(e^{-1})_{ia}\nabla_{v_j}[(e^{-1})_{ja}P]+(\sigma\mathbf{v}+\boldsymbol{\nabla}_{\mathbf{q}}V)_iP\}\\&\quad+\nabla_{q_i}\{-v_iP\}\\&=\sigma_{ij}[T\nabla_{v_i}\nabla_{v_j}P+\nabla_{v_i}\cdot(v_jP)]+\boldsymbol{\nabla}_{\mathbf{q}}V\cdot\boldsymbol{\nabla}_{\mathbf{v}}P-\mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{q}}P.\end{aligned} \tag{A5}$$

The equilibrium solution is

$$P_{\mathrm{eq}}(\dot{\mathbf{q}},\mathbf{q})=\exp\left\{-\frac{1}{T}\left[\tfrac{1}{2}v^2+V(\mathbf{q})\right]\right\}. \tag{A6}$$

If we are only interested in the distribution in $\mathbf{q}$ and not $\mathbf{v}=\dot{\mathbf{q}}$, we may integrate over all possible values of $\dot{\mathbf{q}}$ and obtain Eq. (4).

Finally, I turn to my claim that this system has time-reversal invariant equilibrium correlations $\langle\mathbf{q}(t)\mathbf{q}(0)\rangle$. One way to see this is from the path integral Lagrangian L of Eq. (A3), which can be rewritten as

$$L=\frac{1}{4T}(\ddot{\mathbf{q}}+\boldsymbol{\nabla}V)\sigma^{-1}(\ddot{\mathbf{q}}+\boldsymbol{\nabla}V)+\frac{1}{4T}\dot{\mathbf{q}}\sigma\dot{\mathbf{q}}+L_\eta+\delta(0)\,\mathrm{tr}\ln e+\frac{1}{2T}\partial_t\left(\tfrac{1}{2}\dot{\mathbf{q}}^2+V\right). \tag{A7}$$

As discussed in the main text, the total time derivative at the end can be thrown away if one is only interested in equilibrium correlations. The rest of L is manifestly time-reversal invariant, except possibly the ghost term L_η. (It is again important here that my path integrals are all implicitly defined with symmetric time discretization.)

For this system, the ghost term is actually fairly trivial, as can be seen by integrating out the ghosts. Consider the perturbative expansion of the theory, and imagine an arbitrary ghost loop. In frequency ω, the perturbative ghost propagator is of the form $[-\omega^2-i\sigma(0)\omega+M^2]^{-1}$ (where I have suppressed the indices i,j). The poles in ω all lie in the lower half complex plane. Almost any frequency integral corresponding to a ghost loop can then be seen to vanish by closing the integration contour in the upper half plane. The one exception is for loops with frequency integrals where the integrand does not fall off faster than $1/\omega$ as $|\omega|\to\infty$, since in this case we cannot ignore the contribution of the contour at infinity. This situation arises only for ghost loops with a single ghost propagator and a single vertex of the form $-i\omega\,\delta\sigma(\mathbf{q})$. The resulting frequency integral yields an effective interaction among the $\mathbf{q}$'s of

$$L_\eta\to-\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{-i\omega\,\delta\sigma}{[-\omega^2-i\omega\sigma(0)+M^2]}=-\tfrac{1}{2}\,\delta\sigma. \tag{A8}$$

Resorting indices i,j and not worrying about keeping track of additive constants in the action, this becomes

$$L_\eta\to-\tfrac{1}{2}\mathrm{tr}\sigma. \tag{A9}$$

This is clearly time-reversal invariant.

APPENDIX B: INTEGRATING BY PARTS WITH SYMMETRIC DISCRETIZATION

Consider a term in the action of the form

$$I=\int dt\,\dot{\mathbf{q}}\cdot\boldsymbol{\nabla}f(\mathbf{q}), \tag{B1}$$

which can be naively integrated to yield a boundary term. I will review how this naive integration is justified for the symmetric time discretization used in this paper. In that discretization, I really represents

$$I=\sum_\tau\,(\mathbf{q}_{\tau+1}-\mathbf{q}_\tau)\cdot\boldsymbol{\nabla}f\left(\frac{\mathbf{q}_{\tau+1}+\mathbf{q}_\tau}{2}\right), \tag{B2}$$

where $\tau=t/\Delta t$ is an integer parametrizing the time steps. Now consider the Taylor expansion of $f(\mathbf{q}_{\tau+1})-f(\mathbf{q}_\tau)$ about $\bar{\mathbf{q}}\equiv(\mathbf{q}_{\tau+1}+\mathbf{q}_\tau)/2$:

$$f(\mathbf{q}_{\tau+1})-f(\mathbf{q}_\tau)=\Delta\mathbf{q}\cdot\boldsymbol{\nabla}f(\bar{\mathbf{q}})+O((\Delta q)^3), \tag{B3}$$

where $\Delta\mathbf{q}\equiv\mathbf{q}_{\tau+1}-\mathbf{q}_\tau$. So we can rewrite

$$I=\sum_\tau\,[f(\mathbf{q}_{\tau+1})-f(\mathbf{q}_\tau)+O((\Delta q)^3)]. \tag{B4}$$

In a path integral, contributions to the action survive in the $\Delta t\to0$ limit if they contribute $O(\Delta t)$ or more per time step.[8] The kinetic term determines the size of Δq to be $O((\Delta t)^{1/2})$. Therefore, the $O((\Delta q)^3)$ term in Eq. (B4) can safely be ignored, and what is left trivially cancels between successive time steps, except for boundary terms.

[8]Think about the contributions of a potential term: $\int dt\,U(\mathbf{q})=\Delta t\Sigma_t U(\mathbf{q})$. Such contributions are manifestly $O(\Delta t)$ per time step.

The fact that the error was $O((\Delta q)^3)$ and not $O((\Delta q)^2)$ (which would not be ignorable) in Eq. (B4) depended crucially on the symmetric discretization $\overline{\mathbf{q}}=(\mathbf{q}_{\tau+1}+\mathbf{q}_\tau)/2$.

APPENDIX C: EUCLIDEAN SCHRÖDINGER EQUATION FOR L_2

We can directly determine the equilibrium distribution generated by the generic Lagrangian (28) by transforming the path integral into a Euclidean Schrödinger equation. First, recast the path integral over $\mathbf{q}(t)$ as a path integral over $\mathbf{q}(t)$ and momentum $\mathbf{p}(t)$:

$$Z_2=\lim_{\Delta t\to 0} N'\int\left[\prod_\tau d\mathbf{p}_\tau d\mathbf{q}_\tau\right]e^{-S_2(\mathbf{p},\mathbf{q})}, \tag{C1}$$

$$S_2(\mathbf{p},\mathbf{q})=\sum_\tau\left\{-i\mathbf{p}_\tau\cdot(\mathbf{q}_\tau-\mathbf{q}_{\tau-1})+\Delta t H_2\left(\mathbf{p}_\tau,\frac{\mathbf{q}_\tau+\mathbf{q}_{\tau-1}}{2}\right)\right\}, \tag{C2}$$

$$H_2(\mathbf{p},\mathbf{q})=T\mathbf{p}\sigma^{-1}(\mathbf{q})\mathbf{p}+u(\mathbf{q}), \tag{C3}$$

$$u\equiv U-\frac{1}{\Delta t}\mathrm{tr}\ln e, \tag{C4}$$

which can be checked simply by doing the Gaussian integration over $\mathbf{p}$. In this form, the path integral is well known to correspond to a Schrödinger equation

$$[H_2(\mathbf{p},\mathbf{q})]_{\mathrm{W}}\psi(\mathbf{q},t)=-\dot\psi(\mathbf{q},t), \tag{C5}$$

where the subscript W indicates Weyl ordering of the operators $\mathbf{p}$ and $\mathbf{q}$. The Weyl ordering formula we need in the case at hand is that[9]

$$[p_ip_jA(\mathbf{q})]_{\mathrm{W}}=\frac{1}{4}\{\hat p_i,\{\hat p_j,A(\hat{\mathbf{q}})\}\}, \tag{C6}$$

where I have now introduced hats to emphasize that $\mathbf{p}$ and $\mathbf{q}$ are operators. Taking $\hat{\mathbf{p}}=-i\boldsymbol{\nabla}$, the Schrödinger equation (C5) then becomes

$$\dot\psi=\left\{T\left[(\sigma^{-1})_{ij}\nabla_i\nabla_j+(\sigma^{-1})_{ij,i}\nabla_j+\frac{1}{4}(\sigma^{-1})_{ij,ij}\right]-u\right\}\psi. \tag{C7}$$

I have used the symbol ψ instead of P in this equation because the equilibrium result for ψ must be the square root of the probability distribution. That is, we want the Schrödinger equation (C7) to have

$$\psi_{\mathrm{eq}}=e^{-V(\mathbf{q})/2T} \tag{C8}$$

as its solution rather than $\exp(-V/T)$. To understand this, consider the original Lagrangian L from the discussion of the path integral form of Langevin equations. L differed from L_2 only by a total time derivative. The Schrödinger equation corresponding to L is simply the Fokker-Planck equation (14), which I will now write in the form

$$\dot P=-\hat H_1 P. \tag{C9}$$

Both formulations, in terms of L or L_2, should generate the same equilibrium physics—that is, the same time-dependent correlation functions. A crucial difference between $\hat H_2$ and $\hat H_1$, however, is that the operator $\hat H_2$ is Hermitian, while $\hat H_1$ is not. Now think about what it means to write down a path integral expression for the equilibrium probability distribution $P_{\mathrm{eq}}(\tilde{\mathbf{q}})=\langle\delta(\mathbf{q}-\tilde{\mathbf{q}})\rangle$ in terms of actions that run between arbitrarily large times $-\mathcal{T}$ and $+\mathcal{T}$. The corresponding object in the Schrödinger formulation is

$$P_{\mathrm{eq}}(\tilde{\mathbf{q}})=\lim_{\mathcal{T}\to\infty}\frac{\langle\mathbf{q}(+\mathcal{T})|e^{-\hat H\mathcal{T}}|\tilde{\mathbf{q}}\rangle\langle\tilde{\mathbf{q}}|e^{-\hat H\mathcal{T}}|\mathbf{q}(-\mathcal{T})\rangle}{\langle\mathbf{q}(+\mathcal{T})|e^{-2\hat H\mathcal{T}}|\mathbf{q}(-\mathcal{T})\rangle}, \tag{C10}$$

where $\hat H$ can be either $\hat H_1$ or $\hat H_2$. What dominates the long-time evolution operator $\exp(-\hat H\mathcal{T})$ is the equilibrium state, which I will denote $|\mathrm{eq}\rangle$. The difference between H_1 and H_2 is that the long-time evolution generated by $\hat H_2$ must be symmetric in its overlap with the initial and final states, because $\hat H_2$ is Hermitian. That is, in the large $\mathcal{T}$ limit,

$$\langle\mathbf{q}'|e^{-\hat H_2\mathcal{T}}|\mathbf{q}''\rangle\to\langle\mathbf{q}'|\mathrm{eq}_2\rangle\langle\mathrm{eq}_2|\mathbf{q}''\rangle. \tag{C11}$$

$\hat H_1$ is not Hermitian and so does not have this symmetry. In fact, we know from the usual Fokker-Planck equation corresponding to H_1 that the evolution is dissipative, and the result of long-time evolution is independent of initial conditions. So

$$\langle\mathbf{q}'|e^{-\hat H_1\mathcal{T}}|\mathbf{q}''\rangle\to\langle\mathbf{q}'|\mathrm{eq}_1\rangle. \tag{C12}$$

In the case of H_2, Eq. (C10) then becomes

$$P_{\mathrm{eq}}(\tilde{\mathbf{q}})=|\langle\tilde{\mathbf{q}}|\mathrm{eq}_2\rangle|^2, \tag{C13}$$

whereas for H_1 it becomes

$$P_{\mathrm{eq}}(\tilde{\mathbf{q}})=\langle\tilde{\mathbf{q}}|\mathrm{eq}_1\rangle. \tag{C14}$$

One can now see that the equilibrium wave function represents the square root of P_{eq} in the case of H_2 but P_{eq} itself in the case of H_1.

In any case, we can now uniquely determine $u(\mathbf{q})$, and hence $U(\mathbf{q})$, simply by requiring that the equilibrium amplitude (C8) be a solution of the Schrödinger equation (C7). One finds

$$u=\frac{1}{4T}(\boldsymbol{\nabla}V)\sigma^{-1}(\boldsymbol{\nabla}V)-\frac{1}{2}\boldsymbol{\nabla}(\sigma^{-1}\boldsymbol{\nabla}V)+\frac{T}{4}(\sigma^{-1})_{ij,ij}, \tag{C15}$$

which precisely reproduces the result (25) for the L_2 that describes the Langevin equation, with my claimed result (16) for $\mathbf{F}^{\mathrm{Strat}}$.

[9] See, for example, Ref. [3] for a review of this fact in the present context.

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